An improvement on the maximum number of k-Dominating Independent Sets

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Abstract

Erdős and Moser raised the question of determining the maximum number of maximal cliques or equivalently, the maximum number of maximal independent sets in a graph on n vertices. Since then there has been a lot of research along these lines.

A k-dominating independent set is an independent set D such that every vertex not contained in D has at least k neighbours in D. Let $mi_k(n)$ denote the maximum number of k-dominating independent sets in a graph on n vertices, and let $\zeta_k := \lim_{n\to\infty} \sqrt[n]{mi_k(n)}$. Nagy initiated the study of $mi_k(n)$.

In this article we disprove a conjecture of Nagy and prove that for any even k we have

$$1.489 \approx \sqrt[9]{36} \leq \zeta_k^k.$$

We also prove that for any $k \geq 3$ we have

$$\zeta_k^k \leq 2.053^{\frac{1}{1.053+1/k}} < 1.98,$$

improving the upper bound of Nagy.

Keywords: independent sets, k-dominating sets, almost twin vertices
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1 Introduction

Let $G = G(V, E)$ be a simple graph. For any vertex $v \in V(G)$ let us denote by $d(v)$ the degree of v, $N(v)$ denotes the set of neighbors of v, also called the open neighborhood of v and $N[v]$ denotes the closed neighborhood, i.e. $N[v] := N(v) \cup \{v\}$.

A subset $I \subset V(G)$ is called *independent* if it does not induce any edges. A *maximal independent* set is an independent set which is not a proper subset of another independent set (that is, it cannot be extended to a bigger independent set). A subset $D \subset V(G)$ is a *dominating* set in G if each vertex in $V(G) \setminus D$ is adjacent to at least one vertex of D, that is,

$$\forall v \in V(G) \setminus D : \ |N(v) \cap D| \geq 1.$$

Erdős and Moser raised the question to determine the maximum number of maximal cliques that an n-vertex graph might contain. By taking complements, one sees that it is the same as the maximum number of maximal independent sets an n-vertex graph can have. A dominating and independent set W of vertices is often called a *kernel* of the graph (due to Morgenstern and von Neumann [6]) and clearly, a subset W is a kernel if and only if it is a maximal independent set.

The problem of finding the maximum possible number of kernels has been resolved in many graph families. To state (some of) these results, let $mi_1(n)$ denote the maximum number of maximal independent sets in graphs of order n, and let $mi_1(n, \mathcal{F})$ denote the maximum number of maximal independent sets in the n-vertex members of the graph family $\mathcal{F}$. Answering the question of Erdős and Moser, Moon and Moser proved the following well known theorem.

Theorem 1. *(Moser, Moon, [5]) We have*

$$mi_1(n) = \begin{cases} 3^{n/3} & \text{if } n \equiv 0 \ (mod\ 3) \\ \frac{4}{3} \cdot 3^{\lfloor n/3 \rfloor} & \text{if } n \equiv 1 \ (mod\ 3) \\ 2 \cdot 3^{\lfloor n/3 \rfloor} & \text{if } n \equiv 2 \ (mod\ 3) \end{cases}$$

Moreover, they obtained the extremal graphs. If addition and multiplication by a positive integer denotes taking vertex disjoint union, then Moser and Moon proved that the equality is attained if and only if the graph G is isomorphic to the graph $n/3\ K_3$ (if $n \equiv 0 \pmod 3$); to one of the graphs $(\lfloor n/3 \rfloor - 1)\ K_3 + K_4$ or $(\lfloor n/3 \rfloor - 1)\ K_3 + 2\ K_2$ (if $n \equiv 1 \pmod 3$); $\lfloor n/3 \rfloor\ K_3 + K_2$ (if $n \equiv 2 \pmod 3$).

For the family of connected graphs the analogous question was raised by Wilf [11] and answered by the following result.

Theorem 2. *(Füredi [2], Griggs, Grinstead, Guichard [3]) Let $\mathcal{F}_{con}$ be the family of connected graphs. Then*

$$mi_1(n, \mathcal{F}_{con}) = \begin{cases} \frac{2}{3} \cdot 3^{n/3} + \frac{1}{2} \cdot 2^{n/3} & \text{if } n \equiv 0 \ (mod\ 3) \\ 3^{\lfloor n/3 \rfloor} + \frac{1}{2} \cdot 3^{\lfloor n/3 \rfloor} & \text{if } n \equiv 1 \ (mod\ 3) \\ \frac{4}{3} \cdot 3^{\lfloor n/3 \rfloor} + \frac{4}{3} \cdot 3^{\lfloor n/3 \rfloor} & \text{if } n \equiv 2 \ (mod\ 3) \end{cases}$$

The extremal graphs are determined as well. In these graphs, there is a vertex of maximum degree, and its removal yields a member of the extremal graphs list of the previous theorem.

Wilf [11] and Sagan [10] investigated the case of trees and proved the following theorem.

Theorem 3. *Let $\mathcal{T}$ be the family of trees. Then we have*

$$mi_1(n,\mathcal{T}) = \begin{cases} \frac{1}{2}\cdot 2^{n/2}+1 & \text{if } n \equiv 0 \ (mod\ 2) \\ 2^{\lfloor n/2 \rfloor} & \text{if } n \equiv 1 \ (mod\ 2) \end{cases}$$

Hujter and Tuza determined the maximal number of kernels in triangle free graphs by proving the following result.

Theorem 4. *([4]) Let $\mathcal{T}_\Delta$ be the family of triangle-free graphs. Then for any integer $n \geq 4$ we have*

$$mi_1(n,\mathcal{T}_\Delta) = \begin{cases} 2^{n/2} & \text{if } n \equiv 0 \ (mod\ 2) \\ 5\cdot 2^{(n-5)/2} & \text{if } n \equiv 1 \ (mod\ 2) \end{cases}$$

Other related results can be found in the survey of Chang and Jou [1].

There are lots of variants of domination studied in the literature. A quite natural and often considered one is k-domination. A set D is called *k-dominating* if each vertex in $V(G)\setminus D$ is adjacent to at least k vertices of D. In other words,

$$\forall v \in V(G)\setminus D: \ |N(v)\cap D| \geq k.$$

A k-dominating independent set is called a k-DIS for short. Note that 1-DISes are exactly maximal independent sets. This notion was introduced by Włoch [12]. Nagy [7, 8] addressed the problem of determining the maximum number of k-dominating independent sets (for a given $k \geq 2$) in an n-vertex graph. Generalizing $mi_1(n)$ and $mi_1(\mathcal{F})$ we introduce the following notation.

Notation 5. *For $n,k \geq 1$ let $mi_k(n)$ denote the maximum number of k-DISes in graphs of order n, and let $mi_k(n,\mathcal{F})$ denote the maximum number of k-DISes in an n-vertex graph from the family $\mathcal{F}$. If $\mathcal{F}$ consists of a single graph G, we denote by $mi_k(G)$ the number of k-DISes in G.*

In [8] Nagy proved that for all $k \geq 1$

$$\zeta_k := \lim_{n\to\infty} \sqrt[n]{mi_k(n)}$$

exists. Theorem 1 implies $\zeta_1 = \sqrt[3]{3}$ and, by definition, for $k \geq 2$ we have $\zeta_k \in [1, \sqrt[3]{3}]$. The following upper and lower bounds were established on the values of ζ_k.

Theorem 6. *(Theorem 1.7 [8]) For all $k \geq 3$ we have:*

$$\sqrt{2} \leq \zeta_k^k \leq 2^{\frac{k}{k+1}}.$$

Theorem 7. *(Theorem 1.6 [8]) We have*

$$1.489 \approx \sqrt[9]{36} \leq \zeta_2^2 \leq \sqrt[5]{9} \approx 1.551.$$

Nagy conjectured in [8] (Conjecture 2, p19) that the lower bound of Theorem 6 will be the value of ζ_k^k. Our following theorem disproves this conjecture.

Theorem 8. *For any even k we have*

$$\sqrt[9]{36} \leq \zeta_k^k.$$

Furthermore, $\lim_\infty \zeta_k^k$ exists and is at least $\sqrt[9]{36}$.

In this paper, our aim is to show that there is a constant $\eta > 0$ such that $\zeta_k^k < 2 - \eta$ for all $k \geq 3$, thus improving Theorem 6.

Theorem 9. *For $k \geq 3$ we have*

$$\zeta_k^k \leq 2.053^{\frac{1}{1.053+1/k}} < 1.98.$$

Remark 10. *It is easy to see that $1.98 < 2^{k/(k+1)}$ for $k \geq 588503$. In fact, the following calculation shows that Theorem 9 improves Theorem 6 for all $k \geq 3$. We want to show that*

$$2^{k/(k+1)} > (2+\varepsilon)^{1/(1+\varepsilon+1/k)},$$

for $\varepsilon = 0.053$ and any $k \geq 3$. After rearranging we get

$$2^\varepsilon > (1+\varepsilon/2)^{1+1/k},$$

which is true for $\varepsilon = 0.053$ and $k = 3$. Therefore, it is true for any larger k.

The remainder of the paper is organized as follows. In Section 2 we prove Theorem 8, in Section 3 we prove Theorem 9 and we finish the article with some remarks and open questions in Section 4.

2 Constructions - Proof of Theorem 8

In this section we gather some observations that are related to lower bound constructions. To be more formal, we introduce the following function: let $m(k,t)$ denote the smallest integer n such that there exists a graph on n vertices that contains at least t k-DISes. For our constructions we will need two types of graph products: the *lexicographic product* $G \cdot H$

of two graphs G and H has vertex set $V(G) \times V(H)$ and any two vertices (u, v) and (x, y) are adjacent in $G \cdot H$ if and only if either u is adjacent with x in G or $u = x$ and v is adjacent with y in H.

The *cartesian product* $G \times H$ of two graphs G and H also has vertex set $V(G) \times V(H)$ and any two vertices (u, v) and (x, y) are adjacent in $G \cdot H$ if and only if both u is adjacent with x in G and v is adjacent with y in H.

All our lower bounds follow from the following remark.

Proposition 11. *For any positive integers k, l, t, we have*

(i) $mi_k(n) \geq t^{\lfloor \frac{n}{m(k,t)} \rfloor}$, *and*

(ii) $m(kl, t) \leq lm(k, t)$.

Proof. To prove **(i)** observe that if G is a graph on $m(k, t)$ vertices containing at least t k-DISes, then the graph G' consisting of $\lfloor \frac{n}{m(k,t)} \rfloor$ disjoint copies of G and possibly some isolated vertices, contains at least $t^{\lfloor \frac{n}{m(k,t)} \rfloor}$ many k-DISes. Indeed, all isolated vertices must be contained in every k-DIS of G', and to form a k-DIS of G', one has to pick a k-DIS in every copy of G.

To prove **(ii)** let G be a graph on $m(k, t)$ vertices containing at least t k-DISes. Then, if we denote by E_l the empty graph on l vertices, the graph $G' = G \cdot E_l$ has $lm(k, t)$ vertices and if I is a k-DIS in G, then $I' = \{(u, v) : u \in I\}$ is a (kl)-DIS in G'. $\square$

Proof of Theorem 8. First note (as observed by Nagy already) that $K_3 \times K_3$ contains 6 2-DISes on 9 vertices. Therefore, by **(ii)** of Proposition 11, for every even k we have

$$m(k, 6) \leq \frac{k}{2} m(2, 6) \leq \frac{9k}{2}.$$

Part **(i)** of Proposition 11 yields the statement for even k.

$\square$

Proposition 12. $m(k, 2) = 2k$, $m(k, 3) = 3k$.

Proof. The upper bounds are given by $K_{k,k}$ and $K_{k,k,k}$. For the lower bounds, note that if A and B are two different k-DISes, then we have $|A \setminus B| \geq k$ and $|B \setminus A| \geq k$. Indeed, e.g., if $v \in A \setminus B$ then $N(v)$ must contain at least k vertices in B, while none of these are in A. This observation immediately shows we need at least $2k$ vertices for 2 k-DISes. One can easily see by analyzing possible intersection sizes that it also shows we need at least $3k$ vertices for 3 k-DISes. $\square$

Note that $K_{k,k,\ldots,k}$ gives $m(k, t) \leq tk$. Nagy [8] showed $m(2, 4) = 8$ and $m(2, 6) = 9$.

3 Proof of Theorem 9

First of all we fix $k \geq 3$. Let $\varepsilon = 0.053$ and choose c such that

$$c^k = (2 + \varepsilon)^{\frac{1}{1+\varepsilon+1/k}}.$$

We need to show that $mi_k(n) \leq Ac^n$ for some absolute constant A. We will proceed by induction on n and the base case is covered by a large enough choice of A. Let G be a graph on n vertices containing maximum possible number of k-DISes. We assume that every vertex belongs to at least one k-DIS, as otherwise we can delete the vertex without decreasing the number of k-DISes. Let v be a vertex of minimum degree in G that we denote by δ. Note that we may assume $\delta \geq k$. Indeed, if a vertex v has degree less than k, then it is easy to see that it must be contained in every k-DIS of G. Then it follows that the number of k-DISes in G is at most $mi_k(n - |N(v)| - 1)$ (where $N(v)$ denotes the set of vertices adjacent to v) and we are done by induction.

Consider the following two cases:

Case 1: $\delta \geq (1+\varepsilon)k$.

In this case we use Proposition 5.1 from [8]. Following an inductive argument of Füredi [2], Nagy proved that we have

$$mi_k(n) = mi_k(G) \leq c_0 \max_{\delta \in \mathbb{Z}^+} \{ \left(\frac{k+\delta}{k} \right)^{\frac{n}{\delta+1}} \}.$$

for some universal constant c_0. Let $\delta = (1+\varepsilon')k$. Then we have

$$mi_k(n) \leq c_0 (2+\varepsilon')^{\frac{1}{1+\varepsilon'+1/k}\frac{n}{k}}.$$

By Proposition 14 (see Appendix), the right hand side of the above inequality is monotone decreasing in ε'. Since $\delta \geq (1+\varepsilon)k$, we have $\varepsilon' \geq \varepsilon$. So for fixed $k \geq 3$ we conclude that

$$mi_k(n) \leq c_0 (2+\varepsilon)^{\frac{1}{1+\varepsilon+1/k}\frac{n}{k}} = O(c^n).$$

Case 2: $\delta \leq (1+\varepsilon)k$.

In this case we combine the inductive argument with a new idea. Let v be a vertex of degree δ. The number of k-DISes containing v is at most $mi_k(n - \delta - 1)$ and to bound the number of k-DISes not containing v, we introduce the following auxiliary graph. We say that two non-adjacent vertices x, y of G are *almost twins* if

$$|N(x) \setminus N(y)|,\ |N(y) \setminus N(x)| < k$$

hold. We define T_G to be the graph with vertex set $N(v)$ and x, y form an edge in T_G if they are almost twins in G.

Proposition 13. *If x, y belong to the same connected component in T_G, then they belong to the same k-DISes of G. In particular, they are not connected.*

Proof. It is enough to prove the statement for vertices adjacent in T_G. If x belongs to a k-DIS I with $y \notin I$, then there should be at least k neighbors of y in I and as $x \in I$, we must have $N(x) \cap I = \emptyset$. This implies $|N(y) \setminus N(x)| \geq k$ which contradicts the fact that x and y are almost twins. $\square$

If a pair of vertices $x, y \in N(v)$ belong to different components of T_G then the k-DISes I containing both of x and y are disjoint from $N(x) \cup N(y)$, and $I \setminus \{x, y\}$ should form a k-DIS in $G \setminus (N(x) \cup N(y) \cup \{x, y\})$. As x and y are not almost twins, $|N(x) \cup N(y)| \geq \delta + k$ as wlog. $|N(y) \setminus N(x)| \geq k$ and $|N(x)| \geq k$. Thus, the number of k-DISes containing both of x and y is at most $mi_k(n - \delta - k)$.

On the other hand, if x and y are in the same component C of T_G, then by Proposition 13 any k-DIS I containing both of x and y contains all vertices of C, is disjoint from $N(C)$ and $I \setminus C$ is a k-DIS in $G \setminus (N(C) \cup C)$ and by the second part of Proposition 13 $N(C)$ and C are disjoint. As $|N(C)| \geq \delta$, the number of k-DISes containing both of x and y is at most $mi_k(n - \delta - |C|)$.

Writing $s_1, s_2, \ldots, s_j$ for the sizes of the components of T_G, we obtain

$$mi_k(n) \leq mi_k(n-\delta-1) + \frac{\sum_{i=1}^{j} \binom{s_i}{2} mi_k(n-\delta-s_i) + (\binom{\delta}{2} - \sum_{i=1}^{j} \binom{s_i}{2}) mi_k(n-\delta-k)}{\binom{k}{2}} \tag{1}$$

as every k-DIS I with $v \notin I$ was counted at least $\binom{k}{2}$ times since I must k-dominate v.

Let us choose $B = \beta k$ with $\beta = 0.8$. This implies $2 \leq B \leq k$ as $k \geq 3$. Suppose that in T_G the union of components of size at most B is s. Then the number of pairs of vertices within these components is $\sum_{s_i \leq B} \binom{s_i}{2} \leq \frac{s(B-1)}{2}$. Also, the number of pairs within components of size larger than B is $\sum_{s_i > B} \binom{s_i}{2} \leq \binom{\delta - s}{2}$. Observe that either $s = \delta$ or $s < \delta - B$.

Observe that $mi_k(n - \delta - 2) \geq mi_k(n - \delta - B) \geq mi_k(n - \delta - k)$. Thus majoring all $\binom{\delta}{2}$ summands in the following sum we get:

$$\sum_{i=1}^{j} \binom{s_i}{2} mi_k(n-\delta-s_i) + (\binom{\delta}{2} - \sum_{i=1}^{j} \binom{s_i}{2}) mi_k(n-\delta-k) \leq$$

$$\leq \sum_{s_i \leq B} \binom{s_i}{2} mi_k(n-\delta-2) + \sum_{s_i > B} \binom{s_i}{2} mi_k(n-\delta-B) + (\binom{\delta}{2} - \sum_{i=1}^{j} \binom{s_i}{2}) mi_k(n-\delta-k) \leq$$

$$\leq \frac{s(B-1)}{2} mi_k(n-\delta-2) + \binom{\delta-s}{2} mi_k(n-\delta-B) + \left(\binom{\delta}{2} - \frac{s(B-1)}{2} - \binom{\delta-s}{2}\right) mi_k(n-\delta-k)$$

As $\binom{\delta}{2} = \frac{s(B-1)}{2} + [s(\delta - s) + \frac{s(s-B)}{2}] + \binom{\delta-s}{2}$, this implies that the right hand side of (1) is at most

$$mi_k(n-\delta-1) + \frac{s(B-1)}{2\binom{k}{2}} mi_k(n-\delta-2) + \frac{(s(\delta-s) + \frac{s(s-B)}{2})}{\binom{k}{2}} mi_k(n-\delta-k) + \frac{\binom{\delta-s}{2}}{\binom{k}{2}} mi_k(n-\delta-B). \tag{2}$$

Recall that we want to prove that $mi_k(n) \leq Ac^n$ for some constant A. Using (2), by induction after simplifying it would be enough to show

$$E := c^n - \left[c^{n-\delta-1} + \frac{s(B-1)}{2\binom{k}{2}}c^{n-\delta-2} + \frac{(s(\delta-s) + \frac{s(s-B)}{2})}{\binom{k}{2}}c^{n-\delta-k} + \frac{\binom{\delta-s}{2}}{\binom{k}{2}}c^{n-\delta-B}\right] \geq 0.$$

Using that $k \leq \delta$ and simplifying we obtain

$$\frac{E}{c^{n-\delta-k}} \geq c^{2k} - \left[c^{k-1} + \frac{s(B-1)}{k(k-1)}c^{k-2} + \frac{s(2\delta-s-B)}{k(k-1)} + \frac{(\delta-s)(\delta-s-1)}{k(k-1)}c^{k-B}\right]. \quad (3)$$

We consider two cases, depending on whether s is equal to δ or not. In the latter case, $s < \delta - B$, as noted already.

Case 2.1: $s = \delta$

In this case, the right hand side of (3) simplifies to

$$c^{2k} - c^{k-1} - \frac{\delta(B-1)}{k(k-1)}c^{k-2} - \frac{\delta(\delta-B)}{k(k-1)}.$$

Since δ is at most $(1+\varepsilon)k$ and replacing B by βk, the right hand side of the above inequality is at least

$$c^{2k} - c^{k-1} - (1+\varepsilon)\frac{(\beta k-1)}{(k-1)}c^{k-2} - (1+\varepsilon)(1+\varepsilon-\beta)\left(\frac{k}{k-1}\right) =: f_0(k,\varepsilon,\beta)$$

$$\geq c^{2k} - c^k - (1+\varepsilon)\beta c^k - (1+\varepsilon)(1+\varepsilon-\beta)(1+\frac{1}{1000}) =: f_1(k,\varepsilon,\beta)$$

for $k > 1000$.

Recall that $\varepsilon = 0.053$ and $\beta = 0.8$. Note that the function $a^2 - a - (1+\varepsilon)\beta a - (1+\varepsilon)(1+\varepsilon-\beta)(1+\frac{1}{1000})$ is increasing in the range $a \geq 1$. At $a = (2+\varepsilon)^{\frac{1}{1+\varepsilon+1/1000}}$ the function is positive, thus also for all $k > 1000$ at $a = (2+\varepsilon)^{\frac{1}{1+\varepsilon+1/k}} = c^k$ the function is positive. This means $f_1(k,\varepsilon,\beta) > 0$, which implies $f_0(k,\varepsilon,\beta) > 0$ for $k > 1000$. It is easy to check by a simple computer calculation that $f_0(k,\varepsilon,\beta) > 0$ for $k \leq 1000$ as well.

Case 2.2: $s < \delta - B$.

Note that $\max_{s<\delta-B}\{s(2\delta-s-B)\} < (\delta-B)\delta$. Using this, the right hand side of (3) is at least

$$c^{2k}-c^{k-1}-\frac{(\delta-B)(B-1)}{k(k-1)}c^{k-2}-\frac{(\delta-B)\delta}{k(k-1)}-\frac{\delta(\delta-1)}{k(k-1)}c^{k-B}\geq$$

$$c^{2k}-c^k-(1+\varepsilon-\beta)\frac{(\beta k-1)}{(k-1)}c^k-(1+\varepsilon-\beta)(1+\varepsilon)\frac{k}{k-1}$$

$$-\frac{(1+\varepsilon)(k(1+\varepsilon)-1)}{(k-1)}c^{k-\beta k}:=f_2(k,\varepsilon,\beta)$$

$$\geq c^{2k}-c^k-(1+\varepsilon-\beta)\beta c^k-(1+\varepsilon-\beta)(1+\varepsilon+2/1000)-(1+\varepsilon)(1+\varepsilon(1+1/1000))c^{(1-\beta)k}:=f_3(k,\varepsilon,\beta)$$

for $k>1000$. In the last inequality for bounding the third term we used that $2/1000\geq(1+\varepsilon)/(k-1)$ for $k>1000$ as $\varepsilon=0.053$.

Recall that $\beta=0.8$ and so $1-\beta=\frac{1}{5}$. Observe that the function $a^{10}-a^5-(1+\varepsilon-\beta)\beta a^5-(1+\varepsilon-\beta)(1+\varepsilon+2/1000)-(1+\varepsilon)(1+\varepsilon(1+1/1000))a$ is increasing in a if $a>1$. As for $a=(2+\varepsilon)^{\frac{0.2}{1+\varepsilon+1/1000000}}$ the function is positive, also for all $k>1000000$ for the value $a=(2+\varepsilon)^{\frac{0.2}{1+\varepsilon+1/k}}=c^{k-\beta k}$ the function is positive. This means $f_3(k,\varepsilon,\beta)>0$, which implies $f_2(k,\varepsilon,\beta)>0$ for $k>1000000$. It is easy to check by a simple computer calculation that $f_2(k,\varepsilon,\beta)>0$ for $k\leq 1000000$.

Since $\varepsilon=0.053$ and $c^k=(2+\varepsilon)^{\frac{1}{1+\varepsilon+1/k}}\leq(2+\varepsilon)^{\frac{1}{1+\varepsilon}}$ for any $k\geq 3$, we get $c^k\leq 1.98$ for any $k\geq 3$, completing the proof of Theorem 9.

□

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Appendix

Proposition 14. *Suppose $k \geq 3$ is fixed. Then the function*

$$f(\varepsilon) = (2+\varepsilon)^{\frac{1}{1+\varepsilon+1/k}}$$

is monotone decreasing in ε for $\varepsilon \in [0, \infty)$.

Proof. As f is differentiable, it is enough to prove that the derivative of f is not positive.

$$f'(\varepsilon) = \left(e^{\ln(2+\varepsilon)\frac{1}{1+\varepsilon+\frac{1}{k}}}\right)' = (2+\varepsilon)^{\frac{1}{1+\varepsilon+1/k}} \left(\frac{1}{(2+\varepsilon)(1+\varepsilon+\frac{1}{k})} - \frac{\ln(2+\varepsilon)}{(1+\varepsilon+\frac{1}{k})^2}\right),$$

so as $(2+\varepsilon)^{\frac{1}{1+\varepsilon+1/k}} \geq 0$, it is enough to prove that

$$\frac{1}{(2+\varepsilon)(1+\varepsilon+\frac{1}{k})} - \frac{\ln(2+\varepsilon)}{(1+\varepsilon+\frac{1}{k})^2} \leq 0.$$

Simplifying (and using that $1+\varepsilon+\frac{1}{k} \geq 0$ and $2+\varepsilon \geq 0$), we get

$$1+\varepsilon+\frac{1}{k} \leq (2+\varepsilon)\ln(2+\varepsilon).$$

it is easy to check that for $\varepsilon = 0$ the above inequality holds as $k \geq 3$. Now note that the derivative of the right hand side with respect to ε, namely $1+\ln(2+\varepsilon)$, is larger than the derivative of the left hand side, namely 1. Therefore the above inequality holds for all $\varepsilon \geq 0$, and we are done.

$\square$